AIAA JOURNAL
Vol. 30, No. 6, June 1992

Structural Shape Sensitivity Analysis: Relationship Between Material Derivative and Control Volume Approaches

Jasbir S. Arora,* Tae Hee Lee,† and J. B. Cardoso‡
University of Iowa, Iowa City, Iowa 52242

The material derivative and control volume approaches for structural shape design sensitivity analysis are presented, analyzed, and compared. Starting with a continuum formulation and a general response functional needing sensitivity analysis, the two approaches are derived. It is shown that the final design sensitivity expression for one approach can be obtained from the final expression for the other. Thus, the two approaches are theoretically equivalent. Discretizations of the continuum expressions are presented, and it is shown that the two approaches can lead to different implementations for numerical calculations. A unified interpretation is developed in which the explicit design variations (partial derivatives with respect to the design variables) of the internal and external forces are the major calculations needed to implement the design sensitivity analysis. The discretized forms of the continuum sensitivity expressions are also compared with the ones obtained by starting with the discretized model ab initio. This comparison shows that these two approaches lead to similar expressions for numerical calculations. Therefore, the exact same procedures can be used for computer implementations for both of the approaches. The presented analyses give insights for implementation of the design sensitivity analysis theory with the finite element analysis programs.

Nomenclature

B = strain-displacement matrix
$\bar{B}(V)$ = matrix obtained from differentiation of the shape functions N
b, b_i = design variable
D_{ijkl}, D = material modulus tensor
$D(\,)/Db$ = material design derivative of ( )
dS, ${}^r dS$ = differential surface, and differential surface in the reference domain, respectively
dV, ${}^r dV$ = differential volume, and differential volume in the reference domain, respectively
e_{ij}, e = infinitesimal strain tensor
F = vector of node point internal forces
f_i, f = body force per unit volume
G = integrand of the volume integral in the response function
g = integrand of the displacement specified boundary integral in the response functional
H = twice the mean curvature
h = integrand of the traction specified boundary integral in the response function
I = identity matrix
J, J = Jacobian and Jacobian matrix of the transformation $\Omega \rightarrow {}^b\Omega$, respectively
J_s = area metric of the transformation $\Omega \rightarrow {}^b\Omega$
K = stiffness matrix
N_{ij}, N = matrix of shape functions
n = unit normal vector
P = vector field
Q = unbalanced node point force vector
R = vector of node point external forces
S_u, S_T = displacement and traction specified boundaries, respectively
T_i^0, T^0 = specified surface traction vector
U = node point displacement vector
u_i, u = displacement field
u_i^0 = specified displacement on the boundary S_u
V, ${}^r V$ = material and reference volumes, respectively
V = node point design velocity vector
v_i, v = design velocity field
v_n = normal component of the design velocity field
X = vector of node point coordinate
x_i, x = coordinates of the material point in the original configuration
Z = Jacobian matrix for the transformation $\Omega \rightarrow {}^r\Omega$
δ = arbitrary variation operator
$\bar{\delta}(\,)$ = total design variation of ( ); i.e., $\bar{\delta}(\,) = [D(\,)/Db_k]\delta b_k$
$\bar{\bar{\delta}}(\,)$ = explicit design variation (partial derivative with respect to the design variable) of ( ); i.e., $\bar{\bar{\delta}}(\,) = [\partial(\,)/\partial b_k]\delta b_k$ for which state fields are frozen
$\tilde{\delta}(\,)$ = design variation of the fields that implicitly depend on the design variables, such as displacements, strains, and stresses; also design variation of the functionals with respect to the implicit state fields; for this variation, the explicit dependence on the design variables is frozen
δ_{ij} = Kronecker delta function
ζ, ζ_s = Jacobian and area metric of the transformation $\Omega \rightarrow {}^r\Omega$, respectively
ξ_i = components of intrinsic coordinates; coordinates of a point in the changed configuration for the material derivative approach
σ_{ij}, σ = Cauchy stress tensor
∇ = gradient operator

Subscripts

$,i$ = $\partial(\,)/\partial x_i$, i.e., partial derivative with respect to coordinate x_i
$,b$ = $\partial(\,)/\partial b$, i.e., partial derivative with respect to b

Received Feb. 14, 1991; presented as Paper 91-1214 at the AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, Baltimore, MD, April 8-10, 1991; revision received Sept. 16, 1991; accepted for publication Sept. 19, 1991.

*Professor, Civil and Environmental Engineering and Mechanical Engineering, Optimal Design Laboratory, College of Engineering. Member AIAA.

†Graduate Research Assistant, Optimal Design Laboratory, College of Engineering.

‡Visiting Assistant Professor, Optimal Design Laboratory, College of Engineering; currently Instituto Superior Técnoco, Departmento de Engenharia Mecânica, Universidade, Tecnicade Lisboa, Lisbon, Portugal.

$;i$ = $\partial(\,)/\partial\xi_i$, i.e., partial derivative with respect to the intrinsic coordinate ξ_i

Superscripts

b = quantity measured over the changed domain
r = quantity measured over the fixed reference domain
T = transpose
$'$ = local derivative
$\dot{}$ = material design derivative

I. Introduction

THIS paper analyzes and compares the two basic methods of shape design sensitivity analysis of structural systems: the material derivative approach and the control volume/reference domain approach. In the first approach, the material derivative concept of continuum mechanics is used to obtain variations of the field variables. Also, variations of the volume and surface integrals over a variable domain from the calculus of variations are used to obtain the design sensitivity expression for a response functional. In the control volume approach, all of the quantities and integrals are first transformed to a fixed reference domain. Then variations are taken to develop the design sensitivity expression. The two expressions thus obtained appear to be quite different and can lead to different implementations on the computer. However, both of the approaches can be implemented inside or outside an analysis code for general applications.

Starting with the continuum formulation, the two approaches are presented for linearly elastic structures. In order to compare them, it is important to understand the basic ideas and concepts of the two approaches. Therefore, the basic concepts of material derivative and control volume are explained before presenting the design sensitivity analysis. It is shown that the expression with the material derivative approach can be derived from the final expression obtained with the control volume approach. Thus, the two approaches are theoretically equivalent. Discretization of the two expressions are presented and discussed to study the computational and computer implementation aspects. It is shown that the numerical implementation procedures for the two approaches can be quite different. Both of the procedures have been demonstrated in the literature on several example problems. The procedure based on the control volume approach is interpreted as an extension of the isoparametric concept of finite element analysis to the design sensitivity analysis problem. This viewpoint leads to a very simple explanation of the sensitivity expression, which can aid in computer implementations. An advantage of the material derivative approach is that several different forms of the sensitivity expression can be derived, giving flexibility in terms of computer implementations.

The discrete models for design sensitivity analysis are also analyzed and compared with the continuum models. It is shown that the continuum design sensitivity expression when discretized is similar to the one obtained by starting with a discrete model ab initio. Thus, the continuum and discrete design sensitivity expressions can be implemented on the computer in exactly the same way. An implementation scheme is described that is quite general and simple, needing minimal programing.

Note that only the domain method of design sensitivity analysis with the material derivative approach is analyzed because the boundary integral method with the finite element approach has been determined to be inaccurate in the literature. In addition, only the direct differentiation method is presented, for brevity and clarity, although the adjoint method of design sensitivity analysis can also be discussed. It is understood that all of the discussions and procedures apply to the adjoint method as well.

II. Problem Definition

The problem is to optimize a performance functional for the structure subject to constraints on the response for specified inputs. Most optimization methods require gradients of the response functionals in their iterative calculations. Therefore, we concentrate on this problem of design sensitivity analysis. Considerable work has been done on this problem in the past.[1,2] Both discrete and continuum models have been treated. In the present paper, the continuum formulation is used initially to derive and compare the material derivative and control volume approaches. Then the discretizations are presented and the discrete models analyzed. In the continuum formulation, the virtual work equation (also called a weak form) governing the equilibrium state is given as

$$\int \sigma_{ij}(u)\delta e_{ij}(u)\,\mathrm{d}V - \int f_i\delta u_i\,\mathrm{d}V - \int T_i^0\delta u_i\,\mathrm{d}S_T = 0 \tag{1}$$

where a repeated index implies summation over its range and δu_i is a kinematically admissible virtual displacement field having appropriate smoothness with $\delta e_{ij}(u)$ as the compatible virtual strain field [i.e., $\delta e_{ij}(u) \equiv e_{ij}(\delta u)$].

The strain tensor and its arbitrary variation and the linear stress-strain relationship are given as

$$e_{ij}(u) = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$
$$\delta e_{ij}(u) \equiv e_{ij}(\delta u) = \tfrac{1}{2}(\delta u_{i,j} + \delta u_{j,i}) \tag{2}$$

$$\sigma_{ij} = D_{ijkl}e_{kl} \tag{3}$$

In general, a response functional needing design sensitivity analysis is defined as

$$\psi = \int G(\sigma_{ij}, e_{ij}, u_i, b)\,\mathrm{d}V + \int g(u_i^0, T_i, b)\,\mathrm{d}S_u + \int h(u_i, T_i^0, b)\,\mathrm{d}S_T \tag{4}$$

The functional ψ may represent cost function for the problem or constraints on stresses, strains, displacements, and reaction forces. These constraints may be imposed at a particular point in the structure or over a subdomain.

III. Material Derivative Approach

The material derivative approach of design sensitivity analysis and optimization was formally introduced in Refs. 3 and 4. Substantial development of the method took place during the 1980s. Theory as well as numerical methods have been developed and demonstrated for a variety of problems.[5-14] A more detailed review of the subject can be found in Refs. 1 and 2.

A. Material Derivative Concept

There are a couple of ways to describe the basic material derivative ideas.[5,7,15-19] Here we will present the material

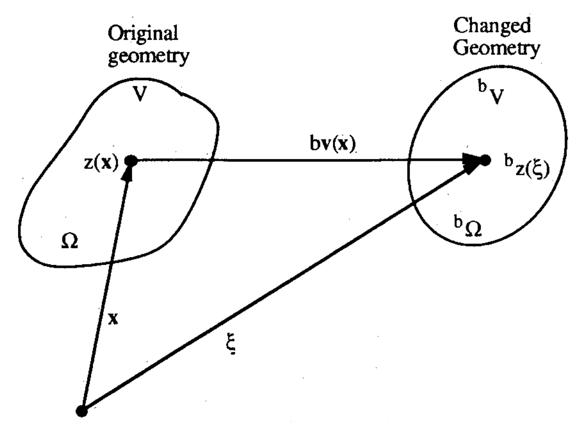

Fig. 1 Variation of domain and a scalar field $z(x)$ using a single parameter family of mappings.

derivative of a quantity with respect to the shape variable as a total derivative consisting of two parts: the first part will consist of the rate of change of the quantity at a fixed point of the domain, and the second part will consist of the changes due to variation of the domain itself. To describe these ideas more clearly, consider a domain Ω occupying a volume V and bounded by the surface S. Let only one parameter b define the change in the shape of the body (other parameters can be treated similarly) and let the changed domain be represented as ${}^b\Omega$, as shown in Fig. 1. Let $z(x,b)$ be a scalar field variable whose total derivative with respect to b at $b = 0$ (the current shape) is desired. Let $\xi_i(x,b)$ be the independent variables in the transformed domain. The functions $\xi_i(x,b)$ can be considered as one-to-one mapping from the domain Ω to ${}^b\Omega$, and the inverse mapping is assumed to exist.

Using the linear Taylor expansion at $b = 0$, the functions $\xi_i(x,b)$ are given as[7]

$$\xi_i(x,b) = x_i + bv_i(x) \tag{5}$$

where $\xi_i(x,0) = x_i$, and $v_i = \partial\xi_i(x,0)/\partial b$ have been used. The quantity $v_i(x)$ is sometimes called the design velocity field. Using the usual definition of a derivative, the total derivative of ${}^bz(x)$ with respect to b at $b = 0$ is then given as

$$\left.\frac{Dz(x,b)}{Db}\right|_{b=0} = \lim_{b\to 0}\frac{{}^bz(\xi) - z(x)}{b}$$

Using Eq. (5) and the linear Taylor expansion for $z(\xi)$ about $b = 0$, the material derivative of $z(x,b)$ becomes

$$\left.\frac{Dz(x,b)}{Db}\right|_{b=0} = \lim_{b\to 0}\frac{{}^bz(x) - z(x)}{b} + \left.\frac{\partial z}{\partial \xi_i}\right|_{b=0}\left.\frac{\partial \xi_i}{\partial b}\right|_{b=0}$$

$$= \left.\frac{\partial z}{\partial b}\right|_{x=\text{fixed}} + \frac{\partial z}{\partial x_i}v_i \tag{6}$$

Using an overdot to represent the material derivative and a prime to represent the local derivative, Eqs. (6) can also be written as

$$\dot{z}(x) = z'(x) + v(x)\bullet\nabla z(x) \tag{7}$$

where $u\bullet v = u^Tv$, and $\nabla = [\partial/\partial x_1\ \partial/\partial x_2\ \partial/\partial x_3]^T$. Equation (7) can also be written as

$$\dot{z}(x) = z'(x) + \bar{z}'(x) \quad \text{with} \quad \bar{z}'(x) = v(x)\bullet\nabla z(x) = v_i z_{,i} \tag{8}$$

where $z_{,i} = \partial z/\partial x_i$. Thus, the material derivative operator that can be used in general is defined as

$$\dot{(\ )} = (\ )' + v\bullet\nabla(\ ) \qquad \text{or} \qquad \dot{(\ )} = (\ )' + v_i(\ )_{,i} \tag{9}$$

Equation (7) or (8) shows that the material derivative of a field variable $z(x)$ consists of two parts. The first part, written as $z'(x)$, is the instantaneous rate of change of z that is observed while standing at a fixed point x of the domain (called the local rate of change or the local derivative of z). The second term $\bar{z}'(x)(= v\bullet\nabla z)$, called the convective part, is the rate of change of z due to a change in the position of the point x that is moving at the design velocity $v(x)$. Figure 2 shows a one-dimensional view of the variations in $z(x)$ based on the two parts of the material derivative of $z(x)$ due to a small change in the domain represented in the parameter b.

It is important to note that most of the field variables in the sensitivity analysis problem are implicit functions of the design variables. It is seen from Eq. (8), however, that the second part of the material derivative (i.e., $\bar{z}'$) is an explicit calculation. That is, once $z(x)$ has been calculated and $v(x)$ is specified, $\bar{z}'$ can be calculated without any further analysis. But the calculation of $z'(x)$ is implicit that needs further analysis and solution of an auxiliary problem. This observation is important in subsequent derivations of design sensitivity expressions.

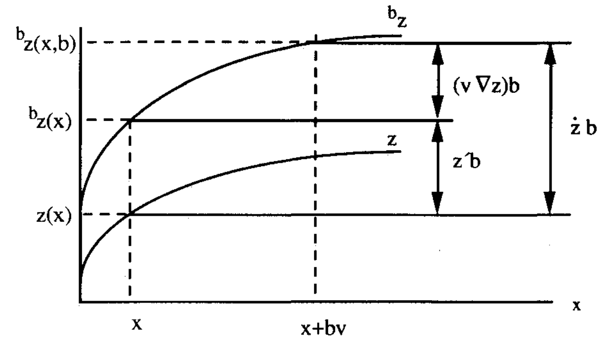

Fig. 2 Description of material derivative of a scalar field.

Using Eq. (5), the Jacobian matrix of the transformation field is defined as $J = [\partial\xi_i/\partial x_j] = I + b\,\nabla v(x)$ and its determinant is written as $J = |J|$. The material derivative of the differential volume can be found in several references[7] and is derived as

$$\dot{\overline{{}^b dV}} = \left.\frac{D({}^b dV)}{Db}\right|_{b=0} = \frac{D}{Db}(J\,dV) = \dot{J}\,dV = v_{k,k}\,dV \tag{10}$$

where $v_{k,k} = \nabla\bullet v$. Similarly, the material derivative of the differential surface area is given as

$$\dot{\overline{{}^b dS}} = \left.\frac{D({}^b dS)}{Db}\right|_{b=0} = \frac{D}{Db}(J_s\,dS) = \dot{J}_s\,dS$$

$$= [\nabla\bullet v - (\nabla v\, n)\bullet n]\,dS$$

$$= (\delta_{kl} - n_k n_l)v_{k,l}\,dS \tag{11}$$

It can also be shown[19,20] that

$$\nabla\bullet v - (\nabla v\,n)\bullet n = \nabla\bullet v - \nabla\bullet(nv_n) + Hv_n \tag{12}$$

where $H = -\nabla\bullet n$ is the curvature of S in R^2 and twice the mean curvature of S in R^3, and $v_n = v\bullet n$ is the normal component of the design velocity field. Therefore, Eqs. (11) give

$$\dot{\overline{{}^b dS}} = [\nabla\bullet v - \nabla\bullet(nv_n) + Hv_n]\,dS \tag{13}$$

The material derivative of the displacement field can be written in the component form or the vector form using the material derivative operator of Eqs. (9) as

$$\dot{u}_i = u_i' + v\bullet\nabla u_i = u_i' + u_{i,j}v_j$$

$$\dot{u} = u' + \nabla u\, v, \qquad \text{where} \qquad \nabla u = [u_{i,j}] \tag{14}$$

The material derivative of the strain tensor can be expressed in several different but equivalent forms.[21] Using the operator in Eqs. (9) directly on $e_{ij}(u)$, its material derivative is given as

$$\dot{\overline{e_{ij}(u)}} = [e_{ij}(u)]' + v\bullet\nabla e_{ij}(u) = e_{ij}(u') + v\bullet\nabla e_{ij}(u) \tag{15}$$

where $\nabla e_{ij}(u)$ is a third-order tensor and $v\bullet\nabla e_{ij}(u)$ is a second-order tensor.

Substituting $u' = \dot{u} - \nabla u\, v$ in Eq. (15) and expanding, the material derivative of the strain field can also be written as

$$\dot{\overline{e_{ij}(u)}} = e_{ij}(\dot{u}) - e_{ij}(\nabla u\, v) + v\bullet\nabla e_{ij}(u) \tag{16}$$

where $e_{ij}(\dot{u})$ and $e_{ij}(\nabla u\, v)$ are defined using the strain operator given in Eq. (2).

Using Eq. (2), the material derivative of the strain tensor can also be written as

$$\overset{\cdot}{\overline{e_{ij}(\boldsymbol{u})}} = \frac{1}{2}\left[\overset{\cdot}{\overline{u_{i,j}}} + \overset{\cdot}{\overline{u_{j,i}}}\right] \tag{17}$$

Using the operator in Eqs. (9), the material derivative of the displacement gradient $u_{i,j}$ is given as

$$\overset{\cdot}{\overline{u_{i,j}}} = u_{i,j}' + \boldsymbol{v}\bullet\nabla u_{i,j} = (u_i')_{,j} + \boldsymbol{v}\bullet\nabla u_{i,j}$$

Substituting $u_i' = \dot{u}_i - \boldsymbol{v}\bullet\nabla u_{i,j}$ and simplifying the resulting expression, we get

$$\overset{\cdot}{\overline{u_{i,j}}} = \dot{u}_{i,j} - u_{i,k}v_{k,j} \tag{18}$$

Substituting Eq. (18) into Eq. (17), we get

$$\overset{\cdot}{\overline{e_{ij}(\boldsymbol{u})}} = e_{ij}(\dot{\boldsymbol{u}}) - \tfrac{1}{2}[u_{i,k}v_{k,j} + u_{j,k}v_{k,i}] \tag{19}$$

Equation (19) can also be obtained directly from Eq. (16).[21]

Equations (15), (16), and (19) give three different forms of the material derivative of the strain tensor. Thus, it is seen that by using these equations different forms of the material derivative of the response functional in Eq. (14) will be obtained.[21] It is important to note that only the first terms in Eqs. (15), (16), and (19) are dependent on the local or total derivative of the displacement field $\boldsymbol{u}$. The other terms in these expressions can be evaluated without any further analysis once $\boldsymbol{u}$ is known and $\boldsymbol{v}$ has been specified. These will be called the explicit design derivatives or design variations.

The material derivative of the stress tensor is obtained from the material derivative of the strain tensor, assuming that D_{ijkl} doesn't depend on the design variables, as

$$\overset{\cdot}{\overline{\sigma_{ij}(\boldsymbol{u})}} = D_{ijkl}\,\overset{\cdot}{\overline{e_{kl}(\boldsymbol{u})}}$$

The material derivative of a volume integral $\psi = \int f\,\mathrm{d}V$ can be found in several references[7] and derived as

$$\dot{\psi} = \frac{\mathrm{D}}{\mathrm{D}b}\left(\int {}^b\!f\,{}^b\mathrm{d}V\right)\bigg|_{b=0} = \frac{\mathrm{D}}{\mathrm{D}b}\left(\int {}^b\!f\,J\,\mathrm{d}V\right)\bigg|_{b=0}$$

$$= \int(\dot{f} + f v_{k,k})\,\mathrm{d}V \tag{20}$$

Similarly, the material derivative of a surface integral $\psi = \int g\,\mathrm{d}S$ is derived as

$$\dot{\psi} = \frac{\mathrm{D}}{\mathrm{D}b}\left(\int {}^b\!g\,{}^b\mathrm{d}S\right)\bigg|_{b=0} = \frac{\mathrm{D}}{\mathrm{D}b}\left(\int {}^b\!g\,J_s\,\mathrm{d}S\right)\bigg|_{b=0}$$

$$= \int(\dot{g} + \dot{J}_s g)\,\mathrm{d}S = \int[\dot{g} + g(\nabla\bullet\boldsymbol{v} - (\nabla\boldsymbol{v}\boldsymbol{n})\bullet\boldsymbol{n})]\,\mathrm{d}S \tag{21}$$

Using Eq. (12), Eqs. (21) can also be written as

$$\dot{\psi} = \int[\dot{g} + g\{\nabla\bullet\boldsymbol{v} - \nabla\bullet(\boldsymbol{n}v_n) + Hv_n\}]\,\mathrm{d}S \tag{22}$$

It can be shown that only the normal component of the design velocity field has an influence on the variation of a surface integral.[3,7] Therefore, Eq. (22) is reduced as

$$\dot{\psi} = \int[\dot{g} + gHv_n]\,\mathrm{d}S \tag{23}$$

B. Design Sensitivity Analysis

In the direct differentiation method, the material derivatives of the state fields are calculated by taking the material derivative of the virtual work equation. Using these, the design variations of any response functional can then be evaluated. Taking the material derivative of the response functional in Eq. (4) and using Eqs. (20) and (23), we get

$$\dot{\psi} = \int(\dot{G} + Gv_{k,k})\,\mathrm{d}V + \int[\dot{g} + gHv_n]\,\mathrm{d}S_u + \int[\dot{h} + hHv_n]\,\mathrm{d}S_T \tag{24}$$

$$\dot{G} = G_{,\sigma_{ij}}\dot{\sigma}_{ij} + G_{,e_{ij}}\dot{e}_{ij} + G_{,u_i}\dot{u}_i + G_{,b} \tag{25}$$

$$\dot{g} = g_{,T_i}\dot{T}_i + g_{,u_i^0}\dot{u}_i^0 + g_{,b} \qquad \dot{h} = h_{,T_i^0}\dot{T}_i^0 + h_{,u_i}\dot{u}_i + h_{,b} \tag{26}$$

To complete the calculations in Eqs. (24–26), we need to evaluate the implicit design derivative terms, such as $\dot{u}_i$, $\dot{e}_{ij}$, $\dot{\sigma}_{ij}$, and $\dot{T}_i$ on S_u. To calculate $\dot{u}_i$, the material derivative of the virtual work equation (1) is taken as

$$\int\overset{\cdot}{\overline{\sigma_{ij}\delta e_{ij}(\boldsymbol{u})}}\,\mathrm{d}V + \int\sigma_{ij}\delta e_{ij}(\boldsymbol{u})\overset{\cdot}{\overline{\mathrm{d}V}} - \int\overset{\cdot}{\overline{f_i\delta u_i}}\,\mathrm{d}V - \int f_i\delta u_i\overset{\cdot}{\overline{\mathrm{d}V}}$$
$$- \int\overset{\cdot}{\overline{T_i^0\delta u_i}}\,\mathrm{d}S_T - \int T_i^0\delta u_i\overset{\cdot}{\overline{\mathrm{d}S_T}} = 0 \tag{27}$$

Expanding and rearranging, Eq. (27) becomes

$$\int\left[\overset{\cdot}{\overline{\sigma_{ij}}}\delta e_{ij}(\boldsymbol{u}) + \sigma_{ij}\overset{\cdot}{\overline{\delta e_{ij}(\boldsymbol{u})}}\right]\mathrm{d}V - \int\left(\dot{f}_i\delta u_i + f_i\overset{\cdot}{\overline{\delta u_i}}\right)\mathrm{d}V$$
$$- \int\left(\dot{T}_i^0\delta u_i + T_i^0\overset{\cdot}{\overline{\delta u_i}}\right)\mathrm{d}S_T = \int\left[f_i\delta u_i - \sigma_{ij}\delta e_{ij}(\boldsymbol{u})\right]\overset{\cdot}{\overline{\mathrm{d}V}}$$
$$+ \int T_i^0\delta u_i\overset{\cdot}{\overline{\mathrm{d}S_T}} \tag{28}$$

Note that δu_i is assumed to depend arbitrarily on design; the form of this dependence will be discussed later. Substituting the material derivative of stresses and strains from Eq. (19) in to Eq. (28) and rearranging, we get

$$\left[\int\sigma_{ij}\delta e_{ij}(\dot{\boldsymbol{u}})\,\mathrm{d}V - \int f_i\overset{\cdot}{\overline{\delta u_i}}\,\mathrm{d}V - \int T_i^0\overset{\cdot}{\overline{\delta u_i}}\,\mathrm{d}S_T\right]$$
$$+ \int\sigma_{ij}(\dot{\boldsymbol{u}})\delta e_{ij}(\boldsymbol{u})\,\mathrm{d}V = \int[f_i\delta u_i - \sigma_{ij}\delta e_{ij}(\boldsymbol{u})]\overset{\cdot}{\overline{\mathrm{d}V}}$$
$$+ \int T_i^0\delta u_i\overset{\cdot}{\overline{\mathrm{d}S_T}} + \int\dot{f}_i\delta u_i\,\mathrm{d}V + \int\dot{T}_i^0\delta u_i\,\mathrm{d}S_T$$
$$+ \frac{1}{2}\int D_{ijkl}(u_{k,m}v_{m,l} + u_{l,m}v_{m,k})\delta e_{ij}(\boldsymbol{u})\,\mathrm{d}V$$
$$+ \frac{1}{2}\int\sigma_{ij}(\delta u_{i,k}v_{k,j} + \delta u_{j,k}v_{k,i})\,\mathrm{d}V$$

Let $\overset{\cdot}{\overline{\delta u_i}}$ be specified as a kinematically admissible field having appropriate smoothness, and, since $\delta e_{ij}(\dot{\boldsymbol{u}})$ is compatible with it, the expression in the first bracket on the left-hand side vanishes because it represents the equilibrium equation for the structure. If we assume δu_i to be independent of design, then $\overset{\cdot}{\overline{\delta u_i}} = 0$ and $\delta e_{ij}(\dot{\boldsymbol{u}}) = 0$. Thus, in this special case again, the terms within the bracket on the left-hand side vanish. Substituting Eqs. (10) and (13) in this equation and rearranging terms, the equation that determines $\dot{\boldsymbol{u}}$ is given as

$$\int\sigma_{ij}(\dot{\boldsymbol{u}})\delta e_{ij}(\boldsymbol{u})\,\mathrm{d}V = \int[(\dot{f}_i + f_i v_{k,k})\delta u_i - \sigma_{ij}\delta e_{ij}(\boldsymbol{u})v_{k,k}]\,\mathrm{d}V$$
$$+ \int(\dot{T}_i^0 + T_i^0 Hv_n)\delta u_i\,\mathrm{d}S_T$$
$$+ \frac{1}{2}\int D_{ijkl}(u_{k,m}v_{m,l} + u_{l,m}v_{m,k})\delta e_{ij}(\boldsymbol{u})\,\mathrm{d}V$$
$$+ \frac{1}{2}\int\sigma_{ij}(\delta u_{i,k}v_{k,j} + \delta u_{j,k}v_{k,i})\,\mathrm{d}V \tag{29}$$

Equation (29) can be used to solve for $\dot{u}_i$ from which $\dot{e}_{ij}$, $\dot{\sigma}_{ij}$, and $\dot{T}_i$ on S_u can be evaluated. Substituting these in Eq. (24), the total derivative of a response functional can be evaluated.

It is important to note that Eq. (29) can be transformed[14] to determine the local derivative of u by substituting $\dot{u} = u' + (\nabla u)v$ into it. Also, using other forms for the material derivative of the stress and strain tensors, other forms for Eq. (29) can be obtained. However, for comparison with the control volume approach, the form in Eq. (29) is more appropriate.

IV. Control Volume Approach

In this approach, all of the material configurations of the deformable body for different designs are separately mapped onto the same conveniently selected fixed domain. These mappings can be viewed as the transformation of the independent variables of the problem. All of the field variables and integrals are transformed to this fixed reference domain before design variations are taken. It is seen that this approach is completely analogous to the isoparametric method of finite element analysis. There, all elements of the same type are mapped onto a parent element of fixed dimensions. Here, all design configurations are mapped onto a fixed reference domain (parent design element). This approach was developed during the 1980s, first for the discrete models[22-24] and then for the continuum models.[25-30] A more detailed review can be found in Refs. 1 and 2.

A. Transformation to the Reference Domain

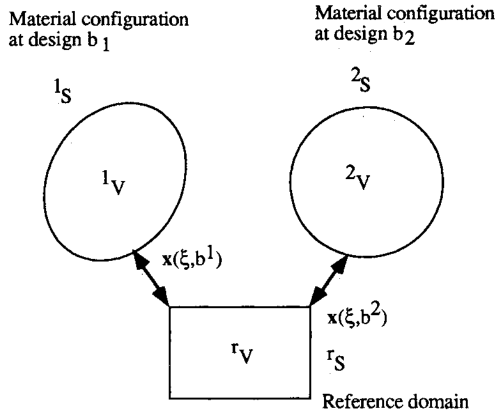

Fig. 3 Transformation to reference domain.

Figure 3 shows the transformation from the material configurations for designs b_1 and b_2 to a fixed domain. This mapping between the material configuration and reference domain can be written in general as $x_i = x_i(\xi, b)$. It can be seen that, as the design changes, this mapping changes with it, but the reference domain never changes. Let ξ_i be the coordinates in the fixed reference domain with volume ${}^rV$ and surface ${}^rS$ that do not change with design variations. In the finite element literature, ξ_i are called the intrinsic, isoparametric, or element's native coordinates. The differential volume and surface elements in the two coordinate systems are related as

$$\mathrm{d}V = \zeta\, {}^r\mathrm{d}V, \qquad \mathrm{d}S = \zeta_s {}^r\mathrm{d}S \tag{30}$$

where $\zeta = |Z|$, $\zeta_s = \|Z^{-T}n\|$, $Z_{ij} = \partial x_i/\partial \xi_j$, and n is the unit outward normal to the fixed reference boundary. The left superscript r indicates quantities in the reference domain.

Substituting the transformations to the reference domain, the virtual work equations (1) and the strain tensor and its arbitrary variation given in Eq. (2) become

$$\int (\sigma_{ij}\delta e_{ij} - f_i \delta u_i)\zeta\, {}^r\mathrm{d}V - \int T_i^0 \delta u_i \zeta_s\, {}^r\mathrm{d}S_T = 0 \tag{31}$$

$$e_{ij}(u) = \tfrac{1}{2}[u_{i;k}\xi_{k,j} + u_{j;k}\xi_{k,i}]$$

$$e_{ij}(\delta u) = \tfrac{1}{2}[\delta u_{i;k}\xi_{k,j} + \delta u_{j;k}\xi_{k,i}] \tag{32}$$

where $\xi_{k,i} = \partial \xi_k/\partial x_i$ and $u_{j;k} = \partial u_j/\partial \xi_k$. Note that the strain tensor and its arbitrary variation in Eqs. (32) now depend explicitly on the design variables through the transformation matrix $\xi_{k,j}$. They also depend implicitly on the design variables because $u_{j;k}$ depends in that way. Transforming the functional in Eq. (4) to the reference domain, we get

$$\psi = \int G(\sigma_{ij}, e_{ij} u_i, b)\zeta\, {}^r\mathrm{d}V + \int g(u_i^0, T_i, b)\zeta_s\, {}^r\mathrm{d}S_u + \int h(u_i, T_i^0, b)\zeta_s\, {}^r\mathrm{d}S_T \tag{33}$$

This functional depends explicitly as well as implicitly on the design variables.

B. Design Sensitivity Analysis

As in the material derivative approach, design variation of the virtual work equation is taken, and after substituting for the design variation of the stress-strain law and strain-displacement relationship, design variation of the displacement field can be computed. Design variations of the stresses, strains, and the response functional ψ can be then computed using the displacement variations.

In the sequel, we shall use the variational notations $\bar{\delta}$, $\bar{\bar{\delta}}$, and $\tilde{\delta}$ to represent the total, explicit, and implicit design variations, respectively. Thus, $\bar{\delta}\psi$ will represent

$$\bar{\delta}\psi = \left(\frac{\mathrm{D}\psi}{\mathrm{D}b}\right)^T \delta b$$

where $\mathrm{D}\psi/\mathrm{D}b$ is the desired design gradient.

The total design variation of the response functional in Eq. (33) gives

$$\bar{\delta}\psi = \int (\bar{\delta}G\zeta + G\bar{\bar{\delta}}\zeta)\, {}^r\mathrm{d}V + \int (\bar{\delta}g\zeta_s + g\bar{\bar{\delta}}\zeta_s)\, {}^r\mathrm{d}S_u + \int (\bar{\delta}h\zeta_s + h\bar{\bar{\delta}}\zeta_s)\, {}^r\mathrm{d}S_T \tag{34}$$

$$\bar{\delta}G = G,_{\sigma_{ij}}\bar{\delta}\sigma_{ij} + G,_{e_{ij}}\bar{\delta}e_{ij} + G,_{u_i}\bar{\delta}u_i + G,_b\delta b \tag{35}$$

$$\bar{\delta}g = g,_{u_i^0}\bar{\delta}u_i^0 + g,_{T_i}\bar{\delta}T_i + g,_b\delta b \tag{36}$$

$$\bar{\delta}h = h,_{u_i}\bar{\delta}u_i + h,_{T_i^0}\bar{\delta}T_i^0 + h,_b\delta b \tag{37}$$

where the fact that ζ and ζ_s depend only explicitly on design has been used. Note that $\bar{\delta}_i^0$ and $\bar{\delta}T_i^0$ are explicit design variations that can be evaluated at the given design. $\bar{\delta}T_i$ on S_u can be evaluated once $\bar{\delta}u_i$ is known. The total design variations of the stress and strain tensors are given as

$$\bar{\delta}\sigma_{ij} = \tilde{\delta}\sigma_{ij} + \bar{\bar{\delta}}\sigma_{ij}, \quad \tilde{\delta}\sigma_{ij} = D_{ijkl}\tilde{\delta}e_{lk}, \quad \bar{\bar{\delta}}\sigma_{ij} = D_{ijkl}\bar{\bar{\delta}}e_{kl} \tag{38}$$

$$\bar{\delta}e_{ij} = \tilde{\delta}e_{ij} + \bar{\bar{\delta}}e_{ij} \tag{39}$$

$$\tilde{\delta}e_{ij} = \tfrac{1}{2}[\bar{\delta}u_{i;k}\xi_{k,j} + \bar{\delta}u_{j;k}\xi_{k,i}] \tag{40}$$

$$\bar{\bar{\delta}}e_{ij} = \tfrac{1}{2}[u_{i;k}\bar{\bar{\delta}}\xi_{k,j} + u_{j;k}\bar{\bar{\delta}}\xi_{k,i}] \tag{41}$$

where $\tilde{\delta}u_i = \bar{\delta}u_i$ is used in Eq. (40) since u_i depends only implicitly on the design variables. In order to calculate $\bar{\delta}u_i$, take total design variation of the virtual work equation (31) as

$$-\int \bar{\delta}(\sigma_{ij}\delta e_{ij}\zeta)\, {}^r\mathrm{d}V + \int \bar{\delta}(f_i \delta u_i \zeta)\, {}^r\mathrm{d}V + \int \bar{\delta}(T_i^0 \delta u_i \zeta_s)\, {}^r\mathrm{d}S_T = 0 \tag{42}$$

Let the virtual displacement δu_i depend arbitrarily on design as before. Expanding Eq. (42), we get

$$\int [\bar{\delta}\sigma_{ij}\delta e_{ij}\zeta + \sigma_{ij}\bar{\delta}(\delta e_{ij})\zeta + \sigma_{ij}\delta e_{ij}\bar{\bar{\delta}}\zeta]\, {}^r\mathrm{d}V - \int [\bar{\bar{\delta}}(f_i\zeta)\delta u_i + (f_i\zeta)\bar{\delta}(\delta u_i)]\, {}^r\mathrm{d}V - \int [\bar{\bar{\delta}}(T_i^0\zeta_s)\delta u_i + (T_i^0\zeta_s)\bar{\delta}(\delta u_i)]\, {}^r\mathrm{d}S_T = 0 \tag{43}$$

Total design variation of the arbitrary strain δe_{ij} is given by

$$\bar{\delta}(\delta e_{ij}) = \bar{\bar{\delta}}(\delta e_{ij}) + \tilde{\delta}(\delta e_{ij}) \tag{44}$$

and use of Eqs. (32) gives

$$\bar{\bar{\delta}}(\delta e_{ij}) = \tfrac{1}{2}[\delta u_{i;k}\bar{\bar{\delta}}\xi_{k,j} + \delta u_{j;k}\bar{\bar{\delta}}\xi_{k,i}] \tag{45}$$

$$\tilde{\delta}(\delta e_{ij}) = \tfrac{1}{2}[\tilde{\delta}(\delta u_{i;k})\xi_{k,j} + \tilde{\delta}(\delta u_{j;k})\xi_{k,i}] \tag{46}$$

Substituting Eq. (44) into Eq. (43), and rearranging and collecting terms, we have

$$\left\{\int \sigma_{ij}\tilde{\delta}(\delta e_{ij})\zeta\, {}^r\mathrm{d}V - \int (f_i\zeta)\tilde{\delta}(\delta u_i)\, {}^r\mathrm{d}V - \int (T_i^0\zeta_s)\tilde{\delta}(\delta u_i)\, {}^r\mathrm{d}S_T\right\}$$
$$+ \int [\bar{\delta}\sigma_{ij}\delta e_{ij}\zeta + \sigma_{ij}\bar{\bar{\delta}}(\delta e_{ij})\zeta + \sigma_{ij}\delta e_{ij}\bar{\bar{\delta}}\zeta]\, {}^r\mathrm{d}V$$
$$- \int \bar{\bar{\delta}}(f_i\zeta)\delta u_i\, {}^r\mathrm{d}V - \int \bar{\bar{\delta}}(T_i^0\zeta_s)\delta u_i\, {}^r\mathrm{d}S_T = 0 \tag{47}$$

Now let $\tilde{\delta}(\delta u_i)$ be specified as a kinematically admissible displacement field having appropriate smoothness. Note from Eq. (46) that $\tilde{\delta}(\delta e_{ij})$ is compatible with $\tilde{\delta}(\delta u_i)$, i.e., $\tilde{\delta}(\delta e_{ij}) = e_{ij}[\tilde{\delta}(\delta \boldsymbol{u})]$. The set $\{\tilde{\delta}(\delta u_i),\ e_{ij}[\tilde{\delta}(\delta \boldsymbol{u})]\}$ can be replaced by any other compatible set, such as $[\delta u_i,\ e_{ij}(\delta \boldsymbol{u})]$. Thus, the expression within the braces in Eq. (47) represents the equilibrium equation for the structure, and so it vanishes. This expression also vanishes if we assume the admissible virtual displacement field δu_i to be independent of design, i.e., $\tilde{\delta}(\delta u_i) = 0$; in this case, $\tilde{\delta}(\delta e_{ij})$ also vanishes. Thus, the total design variation of the virtual work equation in Eq. (47), after using Eqs. (38) for $\bar{\delta}\sigma_{ij}$, and transferring all of the known terms to the right-hand side, becomes

$$\int \tilde{\delta}\sigma_{ij}\delta e_{ij}\zeta\, {}^r\mathrm{d}V = \int \bar{\bar{\delta}}(f_i\zeta)\delta u_i\, {}^r\mathrm{d}V + \int \bar{\bar{\delta}}(T_i^0\zeta_s)\delta u_i\, {}^r\mathrm{d}S_T$$
$$- \int [\bar{\bar{\delta}}\sigma_{ij}\delta e_{ij}\zeta + \sigma_{ij}\bar{\bar{\delta}}(\delta e_{ij})\zeta + \sigma_{ij}\delta e_{ij}\bar{\bar{\delta}}\zeta]\, {}^r\mathrm{d}V \tag{48}$$

This equation can be used to solve for the total design variation $\bar{\delta}u_i$ of the displacement field u_i. Then Eq. (34) can be used to calculate design variation of the response functional ψ.

V. Relationship Between the Methods

It is important to note that the two methods should give the same design sensitivity coefficients if the same assumptions and approximations are used in their calculations. However, the sensitivity equations for the two methods look quite different [e.g., compare Eqs. (24-26) with Eqs. (34-37), and Eq. (29) with Eq. (48)], and so the sequence of calculations and computer implementations can be different. In order to compare the two methods, a correspondence between the variational notation of the control volume approach and the material derivative is established as follows:

$$\bar{\delta}(\ ) = (\dot{\ })\delta b, \qquad \bar{\delta}u_i = \dot{u}_i\delta b, \qquad \bar{\bar{\delta}} = (\ )_{,b}\delta b \tag{49}$$

Using this correspondence, the total design variation of the response functional ψ with the control volume approach given in Eq. (34) can be written as

$$\dot{\psi} = \int (\dot{G}\zeta + G\dot{\zeta})\, {}^r\mathrm{d}V + \int (\dot{g}\zeta_s + g\dot{\zeta}_s)\, {}^r\mathrm{d}S_u$$
$$+ \int (\dot{h}\zeta_s + h\dot{\zeta}_s)\, {}^r\mathrm{d}S_T \tag{50}$$

To obtain the material derivative expressions from the control volume expressions, we need to transform the expressions from the fixed reference domain back to the material configuration. It can be seen that using the correspondence of Eqs. (49), the total design variations of the integrands of ψ given in Eqs. (35) and (37) reduce to the corresponding material derivatives given in Eqs. (25) and (26), respectively. To show the complete correspondence between the two approaches, it needs to be shown that the final sensitivity expression in Eq. (50) can be reduced to Eq. (24), Eq. (48) that determines $\bar{\delta}u_i$ can be reduced to Eq. (29) that determines $\dot{u}_i$, and the total design variation of strains in Eq. (39) can be reduced to the material derivative expression given in Eq. (19).

Note that, in the material derivative approach, the current material configuration is mapped to a moving configuration (refer to Fig. 1) in order to derive the total derivation formulas. In the control volume approach, the fixed reference domain is mapped to the domain occupied by the current material configuration (refer to Fig. 3). This implies that the transformations of the two approaches are not inverse of each other. It can be shown that[19,20]

$$\dot{\zeta} = \zeta v_{k,k}, \qquad \dot{\zeta}_s = \zeta_s H v_n \tag{51}$$

where the velocity field v_k and the curvature H have been defined earlier. Since these are explicit derivatives, we get $\bar{\bar{\delta}}\zeta = \dot{\zeta}\delta b = \zeta v_{k,k}\delta b$ and $\bar{\bar{\delta}}\zeta_s = \dot{\zeta}_s\delta b = \zeta_s H v_n\delta b$. Substituting Eqs. (51) into Eq. (50) and transforming back to the original domain, it is seen that the equation reduces to the form given in Eq. (24) that is obtained using the material derivative approach. It can be further shown[18,20] that

$$\bar{\bar{\delta}}\xi_{i,j} = -\xi_{i,k}v_{k,j}\delta b \tag{52}$$

Using this equation and transforming back to the original domain, the explicit design variation of the strain tensor given in Eq. (41) becomes

$$\bar{\bar{\delta}}e_{ij} = -\tfrac{1}{2}(u_{i,k}v_{k,j} + u_{j,k}v_{k,i})\delta b \tag{53}$$

Using $\tilde{\delta}e_{ij} = e_{ij}(\dot{u})\delta b$ and Eq. (53), the total design variation of the strain tensor given in Eq. (39) becomes

$$\bar{\delta}e_{ij} = [e_{ij}(\dot{u}) - \tfrac{1}{2}(u_{i,k}v_{k,j} + u_{j,k}v_{k,i})]\delta b \equiv \dot{\overline{e_{ij}(u)}}\delta b \tag{54}$$

From Eq. (54), $\dot{\overline{e_{ij}(u)}}$ given in Eq. (19) is recovered.

In order to derive Eq. (29) from Eq. (48), note the following relationships between the design variations and the material derivatives:

$$\tilde{\delta}\sigma_{ij} = \sigma_{ij}(\dot{u})\delta b, \quad \bar{\bar{\delta}}f_i = \dot{f}_i\delta b, \quad \bar{\bar{\delta}}T_i^0 = \dot{T}_i^0\delta b \tag{55}$$

Also, using Eqs. (38) and (53), we have

$$\bar{\bar{\delta}}\sigma_{ij} = -\tfrac{1}{2}D_{ijkl}(u_{k,m}v_{m,l} + u_{l,m}v_{m,k})\delta b \tag{56}$$

Substituting Eqs. (51), (55), and (56) into Eq. (48), we see that Eq. (29) is recovered. Thus, the two approaches are theoretically equivalent.

Note that with the material derivative approach several different forms for Eq. (29) can be obtained for numerical calculations.[7,14,21] In addition, the method can be derived in terms of only the boundary integrals rather than the domain integrals.[7,21] Thus, the design sensitivity analysis with the material derivative approach can be implemented in several alternate but theoretically equivalent ways. This gives a certain amount of flexibility in the numerical calculations, which is an advantage of this method. The control volume approach has not been developed with the boundary integral method, although it can be developed and implemented that way.

VI. Discretization

The design sensitivity expressions of both the approaches—material derivative and control volume—can be discretized for numerical calculations using the standard finite element shape functions. In general, the material derivative expression in Eq. (29) requires specification of the velocity field and calculation of its gradient. In addition, the normal component of the velocity field on the boundary and curvature of the boundary are needed. The control volume approach needs design variation of the Jacobian and the area metric.

Using the isoparametric formulation of finite element analysis, the coordinates and the displacement field are expressed as

$$x = N(\xi)X, \quad \text{or} \quad x_i = N_{ij}X_j$$
$$u = N(\xi)U, \quad \text{or} \quad u_i = N_{ij}U_j \tag{57}$$

where $N(\xi)$ is a matrix of shape functions given in the element's native coordinates ξ_j, and X and U are, respectively, the node point coordinate vector and the node point displacement vector. These quantities have appropriate dimensions depending on the element type. Using the shape functions, the strain vector and its arbitrary variation and the stress vector are given as (note that now the vector notation is used),

$$e = BU, \qquad \delta e = B\delta U, \qquad \sigma = De = DBU \tag{58}$$

where B is the strain-displacement matrix that is calculated by appropriately differentiating N. Substituting appropriate equations into the virtual work Eq. (31) and using the fact that δU is arbitrary, we obtain a finite-dimensional form of the equilibrium equation as

$$Q \equiv R - F = 0 \tag{59}$$

where the equivalent node point external and internal forces are given as

$$R = \int N^T f \zeta^r \mathrm{d}V + \int N^T T^0 \zeta_s{}^r \mathrm{d}S_T \tag{60}$$

$$F = \left[\int B^T D B \zeta^r \mathrm{d}V \right] U \equiv KU \tag{61}$$

A. Control Volume Approach

In order to solve Eq. (48) for $\bar{\delta}u_i$, we introduce the foregoing finite element discretizations (although different discretizations can also be used) for the design variation of the stress vector, strain vector, and its arbitrary variation as

$$\bar{\bar{\delta}}\sigma = \bar{\bar{\delta}}(DB)U, \qquad \tilde{\delta}\sigma = DB\bar{\delta}U, \qquad \bar{\bar{\delta}}(\delta e) = \bar{\bar{\delta}}B\delta U \tag{62}$$

Note that the matrix B depends only explicitly on the design variables. Substituting Eqs. (62) into Eq. (48), we get

$$\int \delta U^T B^T (DB) \bar{\delta} U \zeta^r \mathrm{d}V = \int \delta U^T N^T \bar{\bar{\delta}}(f\zeta)^r \mathrm{d}V + \int \delta U^T N^T \bar{\bar{\delta}}(T^0 \zeta_s)\, \mathrm{d}S_T - \int \delta U^T B^T \bar{\bar{\delta}}(DB) U \zeta^r \mathrm{d}V - \int \delta U^T (\bar{\bar{\delta}} B^T) DBU \zeta^r \mathrm{d}V - \int \delta U^T B^T DBU \bar{\bar{\delta}}\zeta^r \mathrm{d}V \tag{63}$$

Using the facts that δU is arbitrary and the shape function matrix $N(\xi)$ does not depend on the design variables, and rearranging and combining certain terms, Eq. (63) becomes

$$\left[\int B^T D B \zeta^r \mathrm{d}V \right] \bar{\delta} U = \int \bar{\bar{\delta}}(N^T f \zeta)^r \mathrm{d}V + \int \bar{\bar{\delta}}(N^T T^0 \zeta_s)^r \mathrm{d}S_T - \int \bar{\bar{\delta}}(B^T D B) U \zeta^r \mathrm{d}V - \int B^T DBU \bar{\bar{\delta}}\zeta^r \mathrm{d}V$$
$$= \bar{\bar{\delta}} \int N^T f \zeta^r \mathrm{d}V + \bar{\bar{\delta}} \int N^T T^0 \zeta_s{}^r \mathrm{d}S_T - \bar{\bar{\delta}} \int B^T DBU \zeta^r \mathrm{d}V \tag{64}$$

Using Eqs. (59) and (61) in Eq. (64), we have

$$K\bar{\delta}U = \bar{\bar{\delta}}R - \bar{\bar{\delta}}F, \quad \text{or} \quad K\dot{U} = R_{,b} - F_{,b} \tag{65}$$

Equations (65) can now be solved for $\bar{\delta}U$.

To calculate the design sensitivity coefficients of any response functional by the direct differentiation method, we need to evaluate the right-hand side of Eqs. (65). Then using the decomposed structural stiffness matrix saved from the analysis phase, Eqs. (65) can be solved for the derivatives of the displacements. Using these derivatives, the derivatives of the stresses and strains can be evaluated. Substituting the foregoing derivatives into Eq. (34) along with $\bar{\bar{\delta}}\zeta$ and $\bar{\bar{\delta}}\zeta_s$, and carrying out the integration, the design derivative of the response functional can be evaluated.

To implement the foregoing procedure with a finite element program, the element external force generation and internal force recovery routines can be used to calculate the right-hand side of the sensitivity equations (65) (for implementation outside the program, these routines will have to be developed). For this calculation, the displacement that is calculated during the analysis phase is kept fixed as only the explicit derivatives with respect to the design variables need to be calculated. Since analytical derivatives of B, ζ, and ζ_s with respect to the design variables may be difficult to calculate for all types of elements, the central difference approach is suggested to calculate $\bar{\bar{\delta}}R$ and $\bar{\bar{\delta}}F$ in Eq. (65). This approach is quite simple and straightforward to implement, requiring minimal programming. Note that, in order to calculate $\bar{\bar{\delta}}\zeta$ and $\bar{\bar{\delta}}\zeta_s$, relative movement of the finite element mesh due to the change in a boundary node is needed; i.e., the velocity field is needed to perform the foregoing calculation. The finite element mesh generator can be used to calculate this velocity field (however, linearity of the velocity field with respect to the perturbations in design must be maintained[10]), or the unit force/unit displacement approach for an auxiliary structure can be used.[10,12,13] The central difference approach suggested in the foregoing can also be used to calculate $\bar{\bar{\delta}}\zeta$, $\bar{\bar{\delta}}B$, and $\bar{\bar{\delta}}\zeta_s$.

The foregoing approach has been successfully implemented for planar elements with an existing finite element analysis program. Also, analytical differentiation as well as the central difference approach for calculation of $\bar{\bar{\delta}}\zeta$, $\bar{\bar{\delta}}\zeta_s$, $\bar{\bar{\delta}}B$, $\bar{\bar{\delta}}R$, and $\bar{\bar{\delta}}F$ have been implemented. In some implementations, only the elements connected to the design node that was perturbed were used in the calculation of sensitivities. In that case, the design velocity field was not needed, therefore, it was not calculated as in the boundary-layer approach.[5] In the optimal design process with this approach, proper care will have to be exercised to avoid distortion of the finite element mesh. All of these implementations have given quite accurate sensitivity results.[22-24,28,29,31,32]

B. Material Derivative Approach

To implement Eq. (29) with a finite element program, any discretization procedure can be used. However, to compare the material derivative and control volume approaches, we shall use the isoparametric formulation. Accordingly, we need to transform all of the field variables and integrals to an appropriate reference domain and use the discretizations given in Eqs. (57–61). In addition, discretization of the design velocity field and material derivative of the displacement field and some other quantities are given as

$$v = N(\xi)V, \qquad \dot{u} = N(\xi)\dot{U} \tag{66}$$

$$\sigma = De = DBU, \qquad \sigma(\dot{u}) = DB\dot{U} \tag{67}$$

$$\tfrac{1}{2}(\nabla u \nabla v + \nabla v^T \nabla u^T) \equiv \bar{B}(V)U \tag{68}$$

$$\tfrac{1}{2}(\nabla \delta u \nabla v + \nabla v^T \nabla \delta u^T) = \bar{B}(V)\delta U \tag{69}$$

Substituting Eqs. (66–69) into Eq. (29), we have

$$\int \delta U^T B^T DB\dot{U} \zeta^r \mathrm{d}V = \int \delta U^T N^T (\dot{f} + f \nabla \bullet v) \zeta^r \mathrm{d}V + \int \delta U^T N^T (\dot{T}^0 + T^0 H v_n) \zeta_s{}^r \mathrm{d}S_T - \int \delta U^T B^T DBU \nabla \bullet v \zeta^r \mathrm{d}V + \int \delta U^T B^T D\bar{B}U \zeta^r \mathrm{d}V + \int \delta U^T \bar{B}^T DBU \zeta^r \mathrm{d}V \tag{70}$$

Since δU is arbitrary, the discretized equation for evaluation of $\dot{U}$ becomes

$$K\dot{U} \equiv \left[\int B^T D B \zeta\, {}^r\mathrm{d}V\right]\dot{U} = \int N^T(\dot{f} + f\nabla\bullet v)\zeta\, {}^r\mathrm{d}V + \int N^T(\dot{T}^0 + T^0 H v_n)\zeta_s\, {}^r\mathrm{d}S_T - \int B^T D B U \nabla\bullet v \zeta\, {}^r\mathrm{d}V + \int B^T D \bar{B} U \zeta\, {}^r\mathrm{d}V + \int \bar{B}^T D B U \zeta\, {}^r\mathrm{d}V \tag{71}$$

Note that the coefficient matrices in the last two terms in Eq. (71) are transposes of each other. Once the right-hand side of Eq. (71) has been evaluated (a better method to perform this calculation is given later), $\dot{U}$ can be calculated using the previously decomposed stiffness matrix. Derivatives of the strains and stresses can be obtained using $\dot{U}$. Then substituting the foregoing derivatives into Eq. (24) and carrying out the integrations, design derivatives of the response functional can be evaluated. The right-hand side of Eq. (71) can be evaluated once the design velocity field and its gradient, the normal component of the design velocity field on the boundary, curvature of the loaded boundary, design derivatives of the body force, and surface traction have been evaluated.

Details of the implementation of the foregoing procedure for the constant strain triangular (CST) element are given in Ref. 33. This procedure has been used successfully on several example problems.[6-14]

C. Comparison of the Methods

It is seen in the foregoing derivations of the two methods and their discretizations that they need different types of data in the computer implementations. The methods have been successfully implemented in the past using these different approaches.[7,8,11-13,22-24,28-32,34] The quantities needed for implementation of the control volume approach given in Eqs. (63–65) are N, U, D, ζ, ζ_s, B, $\bar{\bar{\delta}}B$, $\bar{\bar{\delta}}f$, $\bar{\bar{\delta}}T^0$, $\bar{\bar{\delta}}\zeta$, and $\bar{\bar{\delta}}\zeta_s$. The quantities needed for implementation of the material derivative approach given in Eq. (71) are N, U, D, ζ, ζ_s, B, H, v, $\nabla\bullet v$, $\bar{B}$, $\dot{f}$, and $\dot{T}^0$. Although these implementations and the data are different, the final sensitivities must be the same if the same models and approximations are used in the two approaches. Thus, we need to develop a unified interpretation of the numerical calculations with the two methods.

The left-hand sides of the discretized equations of the two methods, i.e., Eqs. (65) and (71), are identical. For the control volume approach, Eqs. (65) show that the right-hand side of the sensitivity expression in the continuum form in Eq. (48) represents the difference between the explicit design variations of the externally applied forces and internal forces. This interpretation is difficult to observe from the sensitivity expression in Eq. (29) or (71) that is based on the material derivative approach, though it must be true.[10] The first two integrals in the right-hand side of Eq. (71) represent the explicit design derivatives of the body force and surface traction. These terms can be combined and written as

$$R,_b = \frac{\partial}{\partial b}\left[\int N^T f \zeta\, {}^r\mathrm{d}V + \int N^T T^0 \zeta_s\, {}^r\mathrm{d}S_T\right] \tag{72}$$

Thus, the first two terms in Eq. (71) represent the explicit design derivatives of the node point external forces. Since $\bar{\bar{\delta}}R = R,_b\delta b$, we see that these terms have the same interpretation as for the control volume approach. Based on the foregoing discussion, it is concluded that the last three terms in Eq. (71) must represent the explicit design derivatives of the node point internal forces. Comparing these terms with the corresponding ones in Eq. (63), we observe that the control volume approach needs $\bar{\bar{\delta}}B = B,_b\delta b$ and the material derivative approach needs $\bar{B}$ in the numerical calculations. It will be shown later that $B,_b = -\bar{B}$. To interpret the last three terms of the right-hand side of Eq. (71), let us take the explicit design derivatives (partial derivatives with respect to the design variables) of the node point internal forces defined in Eq. (61) and expand the resulting expression as follows:

$$F,_b = K,_b U = \frac{\partial}{\partial b}\int B^T D B U \zeta\, {}^r\mathrm{d}V = \int B^T D B U \zeta,_b\, {}^r\mathrm{d}V + \int B^T D B,_b U \zeta\, {}^r\mathrm{d}V + \int B,_b^T D B U \zeta\, {}^r\mathrm{d}V = \int B^T D B U \zeta \nabla\bullet v\, {}^r\mathrm{d}V - \int B^T D \bar{B} U \zeta\, {}^r\mathrm{d}V - \int \bar{B}^T D B U \zeta\, {}^r\mathrm{d}V \tag{73}$$

where $\zeta,_b \equiv \dot{\zeta} = \zeta\nabla\bullet v$ has been used. Comparing Eqs. (73) with the last three terms in Eq. (71), we observe that they indeed represent $-F,_b$.

With the foregoing analysis, we see that the sensitivity equations with the material derivative and control volume approaches, i.e., Eqs. (29) and (48) for the continuum form and Eqs. (63) and (71) for the discretized form, can be given identical interpretations.

VII. Analysis of the Continuum and Discrete Approaches

It has been stated in the literature[7,8,13,33] that the continuum approach of the shape design sensitivity analysis does not need differentiation of the element matrices, whereas the discrete approach needs this calculation. To study this aspect, let us derive design sensitivity expressions for the discrete model and compare them with the corresponding expressions obtained by discretizing the continuum expressions.

A. Control Volume Approach

The total design variation of the discretized equilibrium equation (59) gives

$$\bar{\delta}Q \equiv \tilde{\delta}Q + \bar{\bar{\delta}}Q = 0, \qquad \text{or} \qquad -\tilde{\delta}Q = \bar{\bar{\delta}}Q \tag{74}$$

Since

$$-\tilde{\delta}Q = K\tilde{\delta}U, \qquad \bar{\bar{\delta}}Q = \bar{\bar{\delta}}R - \bar{\bar{\delta}}F \tag{75}$$

Eq. (74) reduces to Eq. (65). Thus, the continuum and discrete approaches give the same final sensitivity expression for numerical implementation.

B. Material Derivative Approach

The material derivative of the discretized form of the equilibrium equation (59) gives

$$\dot{Q} = \dot{R} - \dot{F} = 0 \tag{76}$$

Substituting for R and F from Eq. (61) and rearranging, we get

$$K,_b U + K\dot{U} = \int N^T[\dot{f}\zeta + f\dot{\zeta}]\, {}^r\mathrm{d}V + \int N^T[\dot{T}^0\zeta_s + T^0\dot{\zeta}_s]\, {}^r\mathrm{d}S_T$$

where $\dot{K} = K,_b$ has been used. Taking the partial derivative of K given in Eq. (61) with respect to the design variables, substituting it in the above equation, and rearranging the terms, we get

$$K\dot{U} = \int N^T[\dot{f}\zeta + f\dot{\zeta}]\, {}^r\mathrm{d}V + \int N^T[\dot{T}^0\zeta_s + T^0\dot{\zeta}_s]\, {}^r\mathrm{d}S_T - \int B^T D B U \dot{\zeta}\, {}^r\mathrm{d}V - \int B^T D B,_b U \zeta\, {}^r\mathrm{d}V - \int B,_b^T D B U \zeta\, {}^r\mathrm{d}V \tag{77}$$

Using Eqs. (51) in Eq. (77), we observe that it is exactly the same as Eq. (71) if $B,_b = -\bar{B}$. To prove this, let us take the material derivative of the discretized form of the strain vector given in Eqs. (58) as

$$\dot{e} = B\dot{U} + B,_b U \tag{78}$$

where $\dot{B} = B,_b$ has been used. Substituting the finite element discretization into the material derivative expression for the strain tensor given in Eq. (19), and using Eq. (68), we get

$$\dot{e} = B\dot{U} - \bar{B}U \tag{79}$$

Comparing Eqs. (78) and (79), we see that $B,_b = -\bar{B}$. This also gives another way to calculate the matrix $\bar{B}$ introduced in Eq. (68).

C. Comparison and Computer Implementation Aspects

The foregoing analysis shows that the sensitivity expressions obtained by discretizing the continuum models are identical to the ones obtained with the discrete models for both the material derivative and control volume approaches. Therefore, it should be possible to implement them in exactly the same way, i.e., with or without explicitly differentiating the element matrices. The term in question is $K,_b U$ or $\bar{\bar{\delta}}(KU)$ (i.e., $K,_b U \delta b$), which can be written as follows after substituting for K from Eq. (61):

$$K,_b U = \int B,_b^T DBU\zeta^r \mathrm{d}V + \int B^T DB,_b U\zeta^r \mathrm{d}V + \int B^T DBU\zeta,_b{}^r \mathrm{d}V \tag{80}$$

Substituting for $\zeta,_b = \zeta v_{k,k}$ from Eqs. (51) and rearranging Eq. (80), we get

$$K,_b U = \left[\int (B,_b^T DB + B^T D\dot{B},_b + B^T DBv_{k,k})\zeta^r \mathrm{d}V\right] U \tag{81}$$

The expression within the brackets gives $K,_b$. To implement this equation, the matrices B and $B,_b$, ζ and $v_{k,k}$ need to be known. The matrix B is available from the analysis phase, or it can be evaluated by appropriate differentiation of the shape functions N. The matrix $B,_b$ can be evaluated by differentiating B, or it can be evaluated as $B,_b = -\bar{B}$ using Eq. (68). For evaluating $\bar{B}$ from Eq. (68), N, U, and V are needed, which are already known. Thus, Eq. (81) shows that, for the approach where the element matrices are explicitly differentiated, the expression within the square brackets is evaluated first and then postmultiplied by the displacement vector U to complete the calculation.

To develop another way of calculating $K,_b U$, Eq. (80) can be written slightly differently as follows:

$$K,_b U = \int B,_b^T \sigma\zeta^r \mathrm{d}V + \int B^T \bar{\sigma}\zeta^r \mathrm{d}V + \int B^T \sigma\zeta v_{k,k}{}^r \mathrm{d}V \tag{82}$$

where we have used $\sigma = DBU$ and $\bar{\sigma} = DB,_b U$. Combining terms, this equation can be reduced as

$$K,_b U = \int P\zeta^r \mathrm{d}V \tag{83}$$

where

$$P = [B,_b^T \sigma + B^T(\bar{\sigma} + \sigma v_{k,k})] \tag{84}$$

Equation (83) shows that another way to calculate $K,_b U$ is to first calculate the vector P given in Eq. (84) and then carry out the integration. Since the Gaussian quadrature rules are usually used for numerical integration, the vector P needs to be evaluated at only the Gauss points. The data needed to evaluate P are σ, which is available from the analysis phase, D, U, B, $B,_b$, and $v_{k,k}$. This procedure then does not explicitly differentiate the element matrices.

The basic difference between the two numerical procedures to calculate $K,_b U$ given in Eqs. (81) and (83) now becomes transparent: it is the stages at which the displacement vector is used and the integrations are performed. In the first procedure, $K,_b$ is evaluated first by carrying out the integrations and then postmultiplied by U to evaluate the partial derivative of the internal forces at the node points. In the latter procedure, the order of using U and integrations is reversed, i.e., U is used first to evaluate the stresses σ and $\bar{\sigma}$ at the Gauss points, and then the integrations are performed to calculate the partial derivative $K,_b U$ of the node point internal forces KU. Both of the procedures can be implemented inside or outside a finite element analysis program. A count of operations for some typical finite elements shows the second procedure to be requiring fewer numerical operations and fewer integrations. Therefore, that procedure is theoretically more efficient. In actual implementation with a finite element code, the programming environment and the programming effort may dictate as to which procedure is more efficient in an overall sense.

For various types of elements, it may be difficult (and impossible in some cases) to evaluate $B,_b$, $\dot{\zeta}$, $\dot{\zeta}_s$ analytically. Therefore, it is suggested to use a continuum semianalytical approach in which the central finite differences can be used to calculate $K,_b U$ in Eq. (83). In this procedure, the node point forces need to be evaluated at a perturbed design, keeping the node point displacements U unchanged. These forces should be calculated using the equation

$$KU \equiv F = \int B^T \sigma\zeta^r \mathrm{d}V \tag{85}$$

instead of the equation

$$KU \equiv F = \left[\int B^T DB\zeta^r \mathrm{d}V\right] U \tag{86}$$

due to the reasons explained earlier. This central finite difference procedure is independent of the finite element type and can be implemented with minimal programming. A similar procedure is also suggested for calculating the explicit design derivative of the node point external forces.

VIII. Discussion and Conclusions

Two approaches—the material derivative and control volume—for shape design sensitivity analysis of linearly elastic structures are presented and analyzed using the continuum formulation. Only the domain methods where certain quantities are evaluated as volume integrals are analyzed (i.e., the boundary integral approach is not discussed). General discretizations of the final design sensitivity expressions are presented and discussed without reference to a particular type of finite element. The continuum and discrete models for derivation of design sensitivity analysis are also presented and analyzed. Computer implementation aspects are discussed.

Based on the study, the following general conclusions are stated.

1) The material derivative and control volume approaches for structural shape design sensitivity analysis are theoretically identical. The discretized forms for the sensitivity expressions with the two approaches appear to be quite different, needing different data for their numerical implementation. In the past, the approaches have been implemented in quite different ways.[7,8,11-13,29-32,34] The present analyses, however, show that even the discretized forms of the sensitivity expressions for the two methods can be interpreted and implemented in exactly the same way. Thus, a unified viewpoint for the theory and numerical methods of design sensitivity analysis has emerged.

2) Several different forms for the final design sensitivity expression can be derived with the material derivative ap-

proach. This gives flexibility in that the design sensitivity analysis can be implemented in alternate but equivalent ways.

3) An analysis of the design sensitivity expressions obtained with the continuum and discrete forms of the model for the same problem shows that both of the expressions can be interpreted and implemented in exactly the same way if the same finite element discretization procedures are used for both models. The continuum theory, however, gives certain insights that would not be possible with the discrete theory, i.e.,

(i) The discretizations for the analysis model and the design sensitivity analysis model can be different from each other, assuming that both models satisfy appropriate accuracy requirements. This offers flexibilities when the sensitivity analysis is implemented outside an established finite element analysis program.

(ii) The design sensitivity expression with both the approaches can be implemented with or without the explicit differentiation of the element matrices. The procedure that does not evaluate derivatives of the element matrices explicitly needs fewer numerical operations.

4) The control volume approach with the continuum formulation and the discrete form of the design sensitivity analysis for shape and nonshape optimization problems have shown that, for numerical implementation of the sensitivity expressions, one of the major calculations is to evaluate the explicit design variations of the internal and external forces. Analytical procedures can be used for this calculation. However, due to their complexity for general applications, a continuum semianalytical approach is suggested. In this approach it is proposed to use a central finite difference procedure at the element level to evaluate the explicit design variations of the internal and external node point forces from Eqs. (83) and (60), respectively.

5) The proposed semianalytical procedure of the design sensitivity analysis can be implemented inside a finite element program or completely outside of it. Each approach has certain advantages and disadvantages. Implementation inside the program requires minimal programming effort, but complete knowledge of the source code.[23,24,29,32-34] Therefore it is recommended for use by the analysis code developers. Implementation outside the code requires substantial programming effort and may be tedious for certain nonlinear structural problems,[30,32] but offers flexibility in terms of using the same sensitivity analysis package with several different analysis codes. In addition, with this approach, a restart capability is needed for the analysis code so that Eqs. (65) can be solved without complete reanalysis of the structure.

Although the foregoing conclusions are reached by analyzing the direct differentiation method for linearly elastic structures, they are also applicable to the adjoint method. In addition, the analyses can be extended to the nonlinear structural problems, leading to similar conclusions.[20,22-24,26,27,29,30,32,34]

Acknowledgment

This paper is based on part of the research sponsored by the U.S. National Science Foundation under the project "Design Sensitivity Analysis and Optimization of Nonlinear Structural Systems," Grant MSM 89-13218.

References

[1]Adelman, H. M., and Haftka, R. T., "Sensitivity Analysis of Discrete Structural Systems," *AIAA Journal*, Vol. 24, No. 5, 1986, pp. 823-832.

[2]Haftka, R. T., and Adelman, H. M., "Recent Developments in Structural Sensitivity Analysis," *Structural Optimization*, Vol. 1, No. 3, 1989, pp. 137-152.

[3]Zolèsio, J. P., "The Material Derivative (or Speed) Method for Shape Optimization," *Optimization of Distributed Parameter Structures*, edited by E. J. Haug and J. Cea, Sijthoff & Noordhoff, Rockville, MD, 1981.

[4]Choi, K. K., and Haug, E. J., "Shape Design Sensitivity Analysis of Elastic Structures," *Journal of Structural Mechanics*, Vol. 11, No. 2, 1983, pp. 231-269.

[5]Seong, H. G., and Choi, K. K., "Boundary-Layer Approach to Shape Design Sensitivity Analysis," *Mechanics of Structures and Machines*, Vol. 15, No. 2, 1987, pp. 241-267.

[6]Yang, R. J., and Choi, K. K., "Accuracy of Finite Element Based Shape Design Sensitivity Analysis," *Journal of Structural Mechanics*, Vol. 13, No. 2, 1985, pp. 223-239.

[7]Haug, E. J., Choi, K. K., and Komkov, V., *Design Sensitivity Analysis of Structural Systems*, Academic Press, Orlando, FL, 1986.

[8]Choi, K. K., and Seong, H. G., "A Domain Method for Shape Design Sensitivity Analysis of Built-Up Structures," *Computer Methods in Applied Mechanics and Engineering*, Vol. 57, No. 1, 1986, pp. 1-15.

[9]Bennett, J. A., and Botkin, M. E. (eds.), *The Optimum Shape: Automated Structural Design*, Proceedings of International Symposium, Plenum Press, New York, 1986.

[10]Choi, K. K., and Chang, K.-H., "Shape Design Sensitivity Analysis and Optimization of Elastic Solids," *Structural Optimization: Status and Promise*, edited by M. Kamat, Progress in Astronautics and Aeronautics, AIAA, Washington, DC (to be published).

[11]Hou, J. W., Chen, J. L., and Sheen, J. S., "Computational Method for Optimization of Structural Shapes," *AIAA Journal*, Vol. 24, No. 6, 1986, pp. 1005-1012.

[12]Yao, T.-M., and Choi, K. K., "3-D Shape Optimal Design and Automatic Finite Element Regridding," *International Journal for Numerical Methods in Engineering*, Vol. 28, 1989, pp. 369-384.

[13]Belegundu, A. D., and Rajan, S. D., "A Shape Optimization Approach Based on Natural Design Variables and Shape Functions," *Computer Methods in Applied Mechanics and Engineering*, Vol. 66, 1988, pp. 87-106.

[14]Dems, K., and Haftka, R. T., "Two Approaches to Sensitivity Analysis for Shape Variation of Structures," *Mechanics of Structures and Machines*, Vol. 16, No. 4, 1989, pp. 501-522.

[15]Spencer, A. J. M., *Continuum Mechanics*, Longman, London, 1980.

[16]Fung, Y. C., *Foundations of Solid Mechanics*, Prentice-Hall, Englewood Cliffs, NJ, 1965.

[17]Petryk, H., and Mróz, Z., "Time Derivatives of Integrals and Functionals Defined on Varying Volume and Surface Domains," *Archive of Mechanics*, Vol. 38, Nos. 5-6, 1986, pp. 697-724.

[18]Malvern, L. E., *Introduction to the Mechanics of a Continuous Medium*, Prentice-Hall, Englewood Cliffs, NJ, 1969.

[19]Kosinski, W., *Field Singularities and Wave Analysis in Continuum Mechanics*, Halsted Press, New York, 1986.

[20]Lee, T. H., Arora, J. S., and Rim, K., "Design Sensitivity Analysis of Thermoviscoplastic Mechanical and Structural Systems," Optimal Design Lab., College of Engineering, Univ. of Iowa, TR ODL-91.19, Iowa City, IA, Nov. 1991.

[21]Arora, J. S., "An Exposition of Material Derivative Approach for Structural Shape Sensitivity Analysis," Optimal Design Lab., Univ. of Iowa, TR ODL-90.20, Iowa City, IA, 1990; *Computer Methods in Applied Mechanics and Engineering* (submitted for publication).

[22]Ryu, Y. S., Haririan, M., Wu, C. C., and Arora, J. S., "Structural Design Sensitivity Analysis of Nonlinear Response," *Computers and Structures*, Vol. 21, No. 1/2, 1985, p. 245-255.

[23]Haririan, M., Cardoso, J. B., and Arora, J. S., "Use of ADINA for Design Optimization of Nonlinear Structures," *Computers and Structures*, Vol. 26, No. 1/2, 1987, pp. 123-134.

[24]Wu, C. C., and Arora, J. S., "Design Sensitivity Analysis of Nonlinear Response Using Incremental Procedure," *AIAA Journal*, Vol. 25, No. 8, 1987, pp. 1118-1125.

[25]Haber, R. B., "A New Variational Approach to Structural Shape Design Sensitivity Analysis," *Computer-Aided Optimal Design*, edited by C. A. Mota Soares, Vol. 27, Series F: Computer and System Sciences, Springer-Verlag, New York, 1987, pp. 573-587.

[26]Arora, J. S., and Cardoso, J. B., "Design Sensitivity Analysis with Nonlinear Response," *Proceedings of the NASA Symposium on Sensitivity Analysis in Engineering*, NASA CP-2457, Sept. 1986.

[27]Cardoso, J. B., and Arora, J. S., "Variational Method for Design Sensitivity Analysis in Nonlinear Structural Mechanics," *AIAA Journal*, Vol. 26, No. 5, 1988, pp. 595-603.

[28]Phelan, D. G., and Haber, R. B., "Sensitivity Analysis of Linear Elastic Systems Using Domain Parameterization and Mixed Mutual Energy Principle," *Computer Methods in Applied Mechanics and Engineering*, Vol. 77, No. 1/2, 1989, pp. 31-59.

[29]Arora, J. S., and Cardoso, J. B., "A Design Sensitivity Analysis Principle and Its Implementation into ADINA," *Computers and Structures*, Vol. 32, Nos. 3/4, 1989, pp. 691-705.

[30]Tsay, J. J., and Arora, J. S., "Nonlinear Structural Design

Sensitivity Analysis with Path Dependent Response. Part 1: General Theory," *Computer Methods in Applied Mechanics and Engineering,* Vol. 81, No. 2, 1990, pp. 183–208.

[31]Arora, J. S., "Structural Shape Optimization: An Implementable Algorithm," *Electronic Computation,* Proceedings of the 10th Conference, edited by O. Ural and T.-L. Wong, American Society of Civil Engineers, New York, 1991, pp. 213–222.

[32]Arora, J. S., Lee, T. H., and Kumar, V., "Design Sensitivity Analysis of Nonlinear Structures—III: Shape Variation of Viscoplastic Structures," *Structural Optimization: Status & Review,* edited by M. Kamat, Progress in Astronautics and Aeronautics, AIAA, Washington, DC (to be published).

[33]Belegundu, A. D., Rajan, S. D., Choi, B. K., and Budiman, J., "Shape Optimal Design Using Natural Shape Functions," Dept. of Mechanical Engineering, College of Engineering, Pennsylvania State Univ., PSU-ME-86/87-0029, University Park, PA, Aug. 1987.

[34]Poldneff, M. J., Rai, I. S., and Arora, J. S., "Design Variations of Nonlinear Elastic Structures Subjected to Follower Forces," *Computer Methods in Applied Mechanics and Engineering* (submitted for publication).